Homologcal algebra exercise sheet Week 1

1. (a) Show that $\mathbb{Z} \subset \mathbb{Q}$ is epi in the category of rings and that $\mathbb{Q} \subset \mathbb{R}$ is epi in the category of Hausdorff topological spaces.

   (b) Show that in the category of groups, monics are just injective set maps and kernels are monics whose image is a normal subgroup.

2. Show that the functor category $\mathcal{A}^I$ is a category whenever I is small. In that case, show that the *Yoneda embedding*

$$h : I \to \mathbf{Sets}^{I^{op}}, \qquad h_i(j) = \mathrm{Hom}_I(j, i)$$

   is fully faithfull.

3. Let I and $\mathcal{A}$ be categories and assume that every functor $F : I \to \mathcal{A}$ has a limit. Assuming that I is small, show that

$$\lim : \mathcal{A}^I \to \mathcal{A}, \quad F \mapsto \lim_{i \in I} F_i$$

   is a functor and that the universal property of $\lim_{i\in I} F_i$ for all F is equivalent to the assertion that lim is right adjoint to the diagonal functor $\Delta : \mathcal{A} \to \mathcal{A}^I$. Dually, show that colim is left adjoint to Δ.

4. Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be two functors and assume that there are natural transformations $\eta : \mathrm{id}_{\mathcal{A}} \Rightarrow RL$ and $\epsilon : LR \Rightarrow \mathrm{id}_{\mathcal{B}}$ such that the composites

$$L(A) \xrightarrow{L(\eta_A)} LRL(A) \xrightarrow{\epsilon_{L(A)}} L(A) \text{ and } R(B) \xrightarrow{\eta_{R(B)}} RLR(B) \xrightarrow{R(\epsilon_B)} R(B),$$

   are the identity for all objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Show that (L, R) is an adjoint pair of functors.